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B.Sc. Physics Part-III

Paper-V

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In the last lecture we have discussed surface integral. and how to evaluate it for the given problem. Here we discuss volume integral.

For a vector field $\vec{V} = iV_x + jV_y + kV_z$,
and for volume element $d\tau \equiv dx\,dy\,dz$, the Volume integral is given by $\int_v \vec{V} d\tau$

or $\int_v \vec{V} d\tau = i\int_v V_x d\tau + j\int_v V_y d\tau + k\int_v V_z d\tau$.

here $\int_v \vec{V} d\tau = \iiint \vec{V}\, dx\,dy\,dz$.

Note: In place of the vector function $\vec{V}$, ~~one can~~ we can also evaluate volume integral of a scalar function $f(x,y,z)$ (say).

Q.1 Evaluate $\nabla\cdot\vec{V}$ over the unit cube, i.e. $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$. $\vec{V}$ is given by the vector function $ixz - zj + yz^2k$.

Soln.

We have to evaluate $\displaystyle\int_{\text{Unit Cube}} \nabla\cdot\vec{V}\, d\tau$

First, calculate $\nabla\cdot\vec{V} = \dfrac{\partial}{\partial x}(xz) + \dfrac{\partial}{\partial y}(-z) + \dfrac{\partial}{\partial z}(y^2 z)$

~~Here $\vec{V} = ixz - jz + ky$~~

$$\vec{\nabla} \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

$$\vec{V} = ixz - jz + ky^2 z$$

$$\nabla\cdot\vec{V} = z + y^2$$

$$\int \nabla\cdot\vec{V}\, d\tau = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (z+y^2)\, dx\, dy\, dz$$

$$= \int_{y=0}^{1}\int_{z=0}^{1} (z+y^2)\, dy\, dz$$

$$= \int_{y=0}^{1} \left(\frac{1}{2} + y^2\right) dy$$

$$= \frac{1}{2} + \frac{1}{3}$$

$$\boxed{\int_{\text{Unit cube}} \nabla\cdot\vec{V}\, d\tau = \frac{5}{6}}$$

H.W. ① Evaluate $\displaystyle\int_{\text{Unit Cube}} \nabla\cdot\vec{V}\, d\tau$, where $\vec{V}$ is the vector field $ix^2y + jyz + ky^2z$. Integral is over unit cube $0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1$.

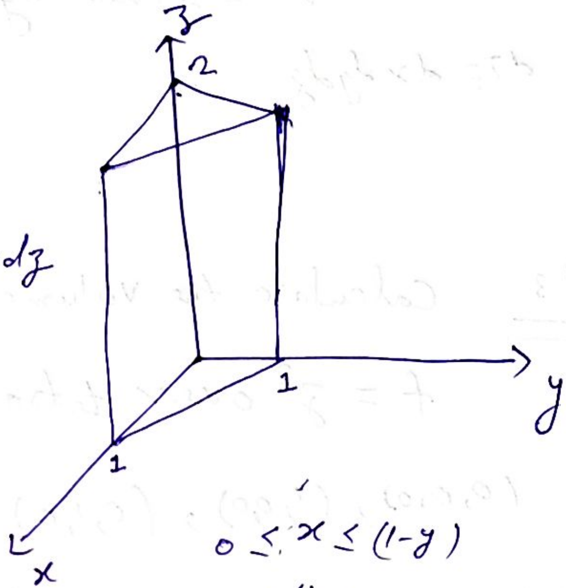
Q.2 Evaluate $\int_v \vec{V}\, d\tau$ over a prism defined by

$0 \le x \le (1-y)$, $0 \le y \le 1$, $0 \le z \le 2$ and vector function $\vec{V}$ is given by $4\hat{i} + 3y^2\hat{j}$. Here $d\tau \equiv dx\,dy\,dz$

Soln.

We need to evaluate

$$\int_v \vec{V}\, d\tau = \int_{x=0}^{1-y}\int_{y=0}^{1}\int_{z=0}^{2} (4\hat{i} + 3y^2\hat{j})\,dx\,dy\,dz$$



$0 \le x \le (1-y)$

$0 \le y \le 1$

$0 \le z \le 2$

Figure shows the prism defined by x, y, z limits.

One can evaluate the three integration in any order. ~~Soln~~ Let us choose to do x integration first

$$\int_v \vec{V}\, d\tau = \int_{y=0}^{1}\int_{z=0}^{2} \left[4x\hat{i} + 3y^2 x\hat{j}\right]_0^{1-y} dy\,dz = \int_{y=0}^{1}\int_{z=0}^{2} \left[4(1-y)\hat{i} + 3y^2(1-y)\hat{j}\right] dy\,dz$$

$$\text{or} \quad \int_v \vec{V}\, d\tau = \int_{z=0}^{2} \left[(4y - 2y^2)\hat{i} + 3\left(\frac{y^3}{3} - \frac{y^4}{4}\right)\hat{j}\right]_0^1 dz$$

$$= \int_{z=0}^{2} \left[2\hat{i} + \frac{1}{4}\hat{j}\right] dz = \left[2z\hat{i} + \frac{z}{4}\hat{j}\right]_0^2 = 4\hat{i} + \frac{1}{2}\hat{j}$$

$$\cancel{\text{or} \int_v \vec{V}\, d\tau = 2\hat{i} + \frac{1}{4}\hat{j}}$$

$$\text{or} \quad \boxed{\int_v \vec{V}\, d\tau = 4\hat{i} + \frac{1}{2}\hat{j}}$$

H.W. 2

Integrate $\int_{v} \vec{V} d\tau$ over the Prism defined by

$0 \le x \le 1, \; 0 \le y \le 3, \; 0 \le z \le (1-x)$. and vector function $\vec{V}$ is given by $x^2\hat{i} + 2y\hat{j} + xz\hat{k}$. Here

$d\tau \equiv dx\, dy\, dz$.

H.W. 3

Calculate the volume integral of the function

$f = z^2$ over tetrahedron with corners at

$(0,0,0),\ (1,0,0),\ (0,1,0),\ (0,0,1)$.

Hint: You have to evaluate $\int_{\text{tetrahedron}} f\, d\tau$. Note that here

f is a scalar function. The process of integration will be same as shown in the Solutions on the previous pages.